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Discrete Mathematics 185 (1998) 275–278

DISCRETE MATHEMATICS

Note

Proof of Melnikov–Vizing conjecture for multigraphs with maximum degree at most 3

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Received 6 August 1997; accepted 11 August 1997

Abstract

Melnikov and Vizing (1997) conjectured that the minimum number of colors sufficient for an edge coloring of mixed multigraph does not exceed either its edge chromatic number or its maximum degree plus one. In this note, this conjecture is proved for multigraphs with maximum degree at most 3. © 1998 Elsevier Science B.V. All rights reserved

1. Introduction

Some problems in Operation Research can be reduced to a special problem of graph coloring, the so-called *mixed coloring*. The term 'mixed coloring' means that the colors are linearly ordered and some arcs of an oriented multigraph can be split into two parts and colored by two colors so that the color of the first part (the tail) does not exceed the color of the second part (the head). A more detailed description of this problem can be found in [3]. It was proved there that such a coloring can be built in polynomial time and the necessary number of colors is equal to the maximum degree of a multigraph.

Melnikov and Vizing [1] modified this problem by dividing the edges of a multigraph into two subsets — links (unoriented edges) and arcs. Each link must be colored by one color while each arc must be divided into two parts and colored by two colors so that the color of the first part is strictly less than the color of the second one.

The formal definitions are as follows.

Let $G=(V;E)$ be a multigraph without loops, and let $E=L\cup A$ where L is the set of links and A is the set of arcs. An *incidentor* or a *semiedge* is a pair (v,e), where $v\in V$, $e\in E$ and v is incident with e. If $e=xy\in A$ then (x,e) is the *initial incidentor*

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[1] This work was partially supported by the grant 96-01-01614 of the Russian Foundation for Fundamental Research.

PII S0012-365X(97)00219-7

or the *tail* of e and (y,e) is the *final incidentor* or the *head* of e (if $e \in L$ then both incidentors are the same). Denote the set of all incidentors by I. A coloring $f: I \to Z_+$ is *proper* if the following conditions are satisfied:

(1) for each $(v,e_1),(v,e_2) \in I$ it holds $f(v,e_1) \neq f(v,e_2)$;

(2) if $e = xy \in A$, then $f(x,e) < f(y,e)$;

(3) if $e = xy \in L$, then $f(x,e) = f(y,e)$.

The *incidentor chromatic number*, $\chi_{la}(G)$, of a multigraph G is the minimum k for which there exists a proper coloring of G in k colors.

Sometimes it is more convenient to use another notation. We shall write $f(e) = (a,b)$ for some arc $e \in A$ if its tail is colored by a and its head is colored by b; for a link $e \in L$ we shall write $f(e) = (a,a)$, or simply $f(e) = a$.

Denote by Δ the maximum degree of a multigraph G and by χ' its edge chromatic number. In [1] it was proved that $\chi_{la} \leqslant \chi' + 1$. The case $L = \emptyset$ was also considered in [1], and it was proved that such a multigraph can be colored in at most $\Delta + 1$ colors. If $A = \emptyset$ then it is the usual edge coloring problem.

Melnikov and Vizing [1] also posed a conjecture that for every multigraph, $\chi_{la} \leqslant \max\{\chi', \Delta + 1\}$.

The main result of this note is that for every multigraph G with $\Delta \leqslant 3$, there exists its proper coloring $f: I \to \{1,2,3,4\}$, and hence each such multigraph satisfies Melnikov–Vizing conjecture.

2. Incidentor coloring of multigraphs with $\Delta \leqslant 2$

Let G be a multigraph with the maximum degree at most 2. Then all its connected components are either cycles or paths. We can consider the case when G consists of cycles only because each path may be converted into a cycle by adding an arc connecting its ends. The term 'cycle' is used with no concern to the orientation of the arcs and means only that each vertex has degree 2.

Lemma 1. *Each cycle C can be colored by three colors. Moreover, if C is not an odd cycle consisting of links only, then there exists a proper coloring of C by colors 1, 2 and 3 such that $f(e) \neq 2$ for each $e \in L$.*

Proof. Let $C = (x_1, x_2, \ldots, x_k)$. If there are no arcs in C then we deal with the edge coloring, and so we can color C by colors $1, 2$ and 3 if C is odd and by colors 1 and 3 if C is even (then the second condition of Lemma 1 will be also true).

So C contains at least one arc. Let it be the arc $e_1 = x_1x_2$. Put $f(e_1) = (2,3)$. Then for $i = 2, 3, \ldots, k-1$ we have two free colors at vertex x_i, so one of them is not equal to 2. If the edge $e_i = x_ix_{i+1}$ is a link then we should color it by this color. If e_i is an arc with the tail (x_i, e_i) then we color it by the least free color and we color the head of e_i by 3; otherwise, the head (x_i, e_i) of e_i is colored by the largest free color and the tail of e_i is colored by 1.

The last edge $e_k = x_k x_1$ can be colored in the same way because the colors 1 and 3 are free at the vertex x_1.

Lemma 1 is proved. □

3. The case of cubic multigraphs with a perfect matching

It is not difficult to show that a cubic graph with a perfect matching can be colored by four colors. But we need a stronger result.

Lemma 2. *Let G be a cubic multigraph with a perfect matching M. Then for every $k \in \{1,2,3,4\}$, there exists a proper coloring of G by the colors* 1, 2, 3 *and* 4, *where all links of M are colored by k.*

Proof. If $k=4$ then we color 2-factor $G \backslash M$ by 1,2 and 3 (by Lemma 1), all links and heads of arcs from M by 4 and all tails of arcs from M by the least free color at its vertex (which is less than 4 because $\Delta = 3$).

The case $k=1$ is similar to the case $k=4$.

If $k=2$ or $k=3$ then we color M, putting $f(e)=k$ for every $e \in M \cap L$ and $f(e)=(2,3)$ for every $e \in M \cap A$. Let $G \backslash M = C_1 \cup \cdots \cup C_m \cup F$ where each C_i, $i=1,2,\ldots,m$ is an odd cycle consisting of links only and F does not contain such cycles. Then by Lemma 1 we can color F by colors 1, 2.5 and 4 so that no link of F is colored by the color 2.5. Then each incidentor (v,e) colored by 2.5 can be recolored by a color in the set $\{2,3\}$ which is free at vertex v (we used only one of these colors at each vertex when we colored the matching M). So, F is colored.

If all edges of M incident with the vertices of C_1 are links then three colors 1, $5-k$ and 4 are free at all vertices of C_1, so it can be easy colored by these three colors. Assume that there is at least one vertex $v \in V(C_1)$ incident with some arc $e \in M$. Denote the neighbors of v in C_1 by u and w. Let for definiteness the incidentor (v,e) be initial. Then $f(v,e)=2$ and the color 3 is free at the vertex v. One of the colors 2 or 3 is also free at vertex u. If 3 is free at the vertex u then we can color the link uv by 3 and the remaining links of C_1 by the colors 1 and 4. If 2 is free at the vertex u then we put $f(uv)=2$, $f(v,e)=1$, $f(vw)=4$, and color the remaining links of the cycle by 1 and 4.

The other odd cycles can be colored in the same way.

Lemma 2 is proved. □

4. The main result

Theorem. *Every multigraph with the maximum degree at most* 3 *can be colored by four colors.*

Proof. Suppose that there are some multigraphs with $\Delta \leqslant 3$ which cannot be colored by four colors. Consider all such multigraphs with the smallest number of vertices, and choose among them the one with the maximum number of edges. Denote it by G. Since G is edge maximal, there exists at most one vertex $v \in V(G)$ whose degree is less than 3. If such v does not exist then put $H = G$. Otherwise, construct a cubic multigraph H as follows: take a copy G' of G and connect the vertices v and v' by the necessary number of edges (arcs or links). By Lemma 2, H has no perfect matching. Then there must be at least one bridge in G, because a cubic multigraph without perfect matchings has at least three bridges [2]. Denote by $e = xy \in E(G)$ a bridge such that the connected component G_0 of a multigraph $G \backslash xy$ which does not contain vertex v is 2-connected. Let for definiteness $x \in V(G_0)$. Then $G_1 = G_0 \backslash \{x\}$ has at least two vertices. So, the multigraph $G_2 = G \backslash G_1$ can be colored by four colors due to the minimality of G.

Denote by k the color of the incidentor (x, e) in this coloring. Now we construct a coloring of G_0 by four colors where the color k is free at the vertex x.

Construct a multigraph H_0 as follows: take a copy G_0' of G_0 and connect the vertices $x \in V(G_0)$ and $x' \in V(G_0')$ by a link. Then multigraph H_0 is cubic with the only bridge x_0x_0'. So it has a perfect matching M'. The link x_0x_0' must belong to M' because both G_0 and G_0' have an odd number of vertices. So, by Lemma 2, H_0 can be colored by four colors where all links of M' are colored by k (and link x_0x_0' too). Hence we obtain a coloring of G_0, in which k is free at the vertex x.

So, our multigraph G is colored by four colors. This contradiction proves the theorem. □

Acknowledgements

The author thanks A.V. Kostochka and O.V. Borodin for their help.

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